Effort Bounded Inspections

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Abstract:

Given an Inspectorate with the task of verifying the adherence of an Operator of a group of facilities to an agreement on permitted activities within those facilities, how large should the inspection effort be and how should it be distributed among the facilities? A game-theoretical approach is described which addresses these important questions, generalizing and extending the applicability of earlier inspection models, which either treated inspection effort as extrinsic, or which imposed special assumptions. A solution of the inspection game, i.e., a Nash equilibrium, is presented in quite general terms, and two applications are presented.

Keywords: game theory, inspection games, resource optimization

1. Problem formulation

The problem of distributing inspection effort across different locations or facilities has been the subject of various analyses in the past. In cases where inspections serve the purpose of deterring an organization or State from violation of an agreement or treaty, game theoretical models involving inspection resource distribution in space and over time have been applied. These models attempt to formulate inspection goals in terms of some objective function such as detection probability, expected time to detection of illegal behaviour, or deterrence.

The allocation of some – continuously divisible – inspection effort was explicitly the subject of analyses by [1], [2] and [3]. The latter work was the stimulus for the present contribution. While Deutsch et al. imposed very specific assumptions, it will be demonstrated that their approach, related to earlier work in [4], can be applied to much more general situations.

Problems of distributing inspection effort across different locations or facilities have also been discussed for some time in the context of applying nuclear material safeguards under the *State Level Approach* by the International Atomic Energy Agency (IAEA) in Vienna. In partial fulfilment of the Non-Proliferation Treaty (NPT) the IAEA verifies the peaceful use of nuclear material in the Treaty's member States [5]. Within this rather general context, the following important questions must be answered: How much inspection effort shall the IAEA allot to a given State? How should that effort be distributed over the individual nuclear facilities within the State? Since major studies along this direction have not been forthcoming from other fields of application, nuclear safeguards in particular and arms control in general have stimulated original work which has become to be known as the field of inspection games; see, e.g., [6] and [7].

In section 2 a general inspection model is developed, that is, a set of assumptions which permits the analysis of the inspection problem in quantitative terms. In section 3 a Nash equilibrium of the resulting non-cooperative two-person game is presented. Since the game and its solution are expressed in rather general terms, two applications are given in sections 4 and 5. The concluding section 6 remarks on further applications and future extensions of the results are discussed.

2. The Model

We consider K facilities operated by an organization or State which are subject, under a verification agreement, to control by an Inspectorate or Inspector. An illegal activity, in the sense of the agreement, is assumed to take place *in at most one facility,* since this way the number of staff members involved in the illegal activity is kept small.

If the illegal activity is performed in facility i, $i = 1, \dots, K$, then it will be detected with some probability p_i. Note that the p_i, $i = 1, \dots, K$, do not necessarily sum to unity. For example, if an a-priori required detection probability of $p_i = 1/2$ is assumed in all facilities, then for $K \geq 3$ the sum of the p_i's exceeds 1.

At IAEA and other inspection authorities the inspection measures z_i spent at the i-th facility, $i = 1, \dots, K$, depend on the a-priori required detection probabilities p_i for any of the facilities, i.e., $z_i(p_i)$. For example, $p_i = 0.9$ for 'high' and $p_i = 0.2$ for 'low' probability level; see [5]. Since however, values of this kind can hardly be justified by formal means, we consider in this paper the reverse case: At the i-th facility, the inspection measure z_i is taken which then results in a detection probability p_i for that facility, i.e. $p_i(z_i)$, under the condition that the illegal activity takes place in facility i. Note that the $p_i(z_i)$ are conditional probabilities which do not need to add up to one when summing with respect to the conditional event (see the example above). The probabilities $p_i(z_i)$ are related to each other only via the effort boundary condition; see (1) below. Also note that according to $(z_1, \dots, z_K)$, in any of the facilities with $z_i > 0$ an inspection is performed by applying the inspection measures $z_i, i = 1, \dots, K$. Assumptions on $p_i(z_i)$ are given in (7).

Also assume that the unit inspection measure in the i-th facility requires the inspection effort w_i (> 0), $i = 1, \dots, K$ and that the total available inspection effort is fixed,

$$\sum_{i=1}^{K} w_i z_i = w \, .$$

Therefore, the Inspector's strategy set is

$$Z := \left\{ \boldsymbol{z} = (z_1, \dots, z_K) : 0 \leq z_i, i = 1, \dots, K, \sum_{i=1}^{K} w_i z_i = w \right\}. \quad (1)$$

Let q_i, $i = 1, \dots, K$, be the probability that the illegal activity takes place in the i-th facility and let q_0 be the probability for legal behaviour. These probabilities sum to unity, because as stated above, the Operator will act illegally in at most one facility. Therefore, the Operator's strategy set is

$$Q := \left\{ \boldsymbol{q} = (q_0, q_1, \dots, q_K) : 0 \leq q_i \leq 1, \; i = 0, \dots, K, q_0 + \sum_{i=1}^{K} q_i = 1 \right\}. \quad (2)$$

In a non-cooperative two-person game formulation of this inspection problem, the payoffs to the Inspector (player 1) and to the Operator (player 2) are given by

$$\begin{array}{ll} (-a_i, -b_i) & \text{for detected illegal activity of the Operator in facility } i \\ (-c_i, d_i) & \text{for undetected illegal activity of the Operator in facility } i \\ (0,0) & \text{for legal behavior of the Operator in all facilities} \end{array} \quad , \quad (3)$$

where we have for all $i = 1, \dots, K$

$$0 < a_i < c_i \quad \text{and} \quad 0 < Min(b_i, d_i) \, .$$

Note that $a_i > 0$ since the highest priority of the Inspector is to deter the Operator from illegal behaviour.

By (3), the expected payoff to both players, conditional on the facility i, $i = 1, \dots, K$, at which the illegal activity is performed, is, for all $i = 1, \dots, K$, given by

$$\begin{aligned} &-a_i p_i(z_i) - c_i(1 - p_i(z_i)) \quad \text{and} \\ &-b_i p_i(z_i) + d_i(1 - p_i(z_i)). \end{aligned} \quad (4)$$

Define for all $i = 1, \dots, K$

$$A_i := b_i + d_i \quad \text{and} \quad B_i := c_i - a_i \, . \quad (5)$$

Because the Operator behaves illegally in at most one facility (see above), and because the probability of behaving illegally in facility i is q_i, the (unconditional) expected payoffs to both players are, using (4) and (5), for all $\boldsymbol{z} \in Z$ and for all $\boldsymbol{q} \in Q$ given by

$$\begin{aligned} In(\boldsymbol{z}, \boldsymbol{q}) &:= \sum_{i=1}^{K} (-c_i + B_i p_i(z_i)) q_i \quad \text{and} \\ Op(\boldsymbol{z}, \boldsymbol{q}) &:= \sum_{i=1}^{K} \left(d_i - A_i p_i(z_i)\right) q_i \, . \end{aligned} \quad (6)$$

By (1), (2) and (6), a non-cooperative two-person game (Z, Q, In, Op) is defined.

The functional dependence $p_i(z_i)$ is assumed to be strictly monotonically increasing and strictly concave, i.e., for $z_i \geq 0$ and for all $i = 1, \dots, K$,

$$\begin{aligned} &0 \leq p_i(z_i) \leq 1, \quad p_i(0) = 0, \qquad \text{and} \\ &\frac{dp_i(z_i)}{dz_i} > 0 \qquad \frac{d^2 p_i(z_i)}{dz_i^2} < 0 \, . \end{aligned} \quad (7)$$

Justification of (7): One can assume reasonably that the higher the inspection measures z_i in facility i, the higher the conditional detection probability $p_i(z_i)$. Thus, $p_i^{-1}(z_i)$ must be monotone increasing. The strictly monotone behaviour of $p_i(z_i)$ assures the existence of its inverse $p_i^{-1}(z_i)$, i.e., $p_i\left(p_i^{-1}(z_i)\right) = p_i^{-1}(p_i(z_i)) = z_i$ for all $z_i \geq 0$ and all $i = 1, \dots, K$. The strict concavity of $p_i(z_i)$ is needed to assure a global maximum of the Inspector´s expected payoff; see the proof of the Theorem.

In chapter 6 of [4] an inspection model is considered which is, with respect to modelling, very different to the inspection model described in this paper, but its game theoretical solution, i.e., the Nash equilibrium, is a special case of the game theoretical solution of the inspection game presented in this paper. Therefore, for reasons of comparisons, we present the inspection game of chapter 6 of [4] in some detail.

The Inspector chooses the facility in which the inspection is performed with probability z_i, $i = 1, \dots, K$, and only one facility is inspected. Therefore, the Inspector's strategy set in [4] is given by

$$\tilde{Z} := \left\{ \boldsymbol{z} = (z_1, \dots, z_K): 0 \leq z_i, i = 1, \dots, K, \sum_{i=1}^{K} z_i = 1 \right\}.$$

If we put $w = w_i = 1$ for all $i = 1, \dots, K$, then we have, using (1), $\tilde{Z} = Z$ and the Inspector's strategy sets coincide. To illustrate the difference in the meaning of z_i in both inspection models, consider $K = 3$ facilities with $w = w_1 = w_2 = w_3 = 1$ and assume that w and w_i are measured in hours. If the Inspector plays $\boldsymbol{z} = (z_1, z_2, z_3) = (1/3, 1/3, 1/3)$, then he performs in the inspection model described in this paper an inspection in all three facilities each one lasting 20 minutes. In the inspection model in [4], however, only one of the three facilities is inspected, and each one is selected with probability $1/3$.

In [4], the Operator chooses the facility in which the illegal activity will take place with probability q_i, $i = 1, \dots, K$, whereby only illegal behaviour is considered. Thus, the Operator's strategy set in [4] coincides with the Operator's strategy set (2) if we assume $q_0 = 0$ in (2).

Regarding the payoffs to both players, in [4] it is assumed that if the inspection is performed in the same facility in which the illegal activity takes place, then detection happens with detection probability $1 - \beta_i$, $i = 1, \dots, K$. In [4] it is shown that the (unconditional) expected payoffs to both players are then given by

$$\begin{aligned} \widetilde{In}(\boldsymbol{z}, \boldsymbol{q}) &= \sum_{i=1}^{K} (-c_i + B_i(1 - \beta_i) z_i) q_i \quad \text{and} \\ \widetilde{Op}(\boldsymbol{z}, \boldsymbol{q}) &= \sum_{i=1}^{K} (d_i - A_i(1 - \beta_i) z_i) q_i \ , \end{aligned} \tag{8}$$

where A_i and B_i are given by (5). Comparing the payoffs (6) and (8) we see that if we put $p_i(z_i) = (1 - \beta_i) z_i$ for all $i = 1, \dots, K$, then the payoffs in (6) simplify to the payoffs in (8). Therefore, the inspection model in this paper is a far more general inspection model than that in chapter 6 of [4], and in case of $p_i(z_i) = (1 - \beta_i) z_i$ and $w = w_i = 1$ for all $i = 1, \dots, K$, both inspection games formally coincide. However, as mentioned above, both inspection models describe very different inspection problems, and it is surprising that they lead to the same forms for the expected payoffs, which then result in corresponding Nash equilibria.

Note that in [2] and [4] the special case $p_i(z_i) = z_i$, $i = 1, \dots, K$, is also considered in a $(K + 1)$ -person game with K independently acting Operators, each responsible for one facility only. We will come back to this model in section 6.

3. Nash Equilibria

In this section we solve the non-cooperative two-person game (Z, Q, In, Op) by determining the so-called Nash-equilibrium; see [8]. A Nash equilibrium is a pair of strategies with the property that unilateral deviation of one player from its equilibrium strategy does not improve the deviator's payoff. Formally, the pair of strategies $(\boldsymbol{z}^*, \boldsymbol{q}^*)$ with $\boldsymbol{z}^* \in Z$ and $\boldsymbol{q}^* \in Q$ constitutes a Nash equilibrium of the game (Z, Q, In, Op) if and only if the Nash the equilibrium conditions

$$\begin{aligned} In^* &:= In(\boldsymbol{z}^*, \boldsymbol{q}^*) \geq In(\boldsymbol{z}, \boldsymbol{q}^*) \quad \text{for all} \quad \boldsymbol{z} \in Z \\ Op^* &:= Op(\boldsymbol{z}^*, \boldsymbol{q}^*) \geq Op(\boldsymbol{z}^*, \boldsymbol{q}) \quad \text{for all} \quad \boldsymbol{q} \in Q \end{aligned} \tag{9}$$

are fulfilled. Because of (7), the existence of a Nash equilibrium for the game (Z, Q, In, Op) can be assured using the Theorem by Nikoida-Isoda; see [9].

The Nash equilibrium of the game (Z, Q, In, Op) is given in

Theorem. Given the non-cooperative inspection game (Z, Q, In, Op) and assume that (7) is fulfilled. Without loss of generality assume

$$d_1 > d_2 > \cdots > d_K > d_{K+1} =: 0 \,. \tag{10}$$

Let $1 \leq k \leq K + 1$ be chosen so that

$$\sum_{i=1}^{k-1} w_i p_i^{-1} \left(\frac{d_i - d_k}{A_i} \right) < w \leq \sum_{i=1}^{k} w_i p_i^{-1} \left(\frac{d_i - d_{k+1}}{A_i} \right) \tag{11}$$

where $p_i^{-1}()$ is the inverse function of $p_i()$ for

$i = 1, \ldots, K$, and the first inequality is to be ignored for $k = 1$ and the second for $k = K + 1$.

(i) For $1 \le k \le K$ equilibrium strategies for the two players are

$$z_i^* = \begin{cases} p_i^{-1}\left(\frac{d_i - Op^*}{A_i}\right) & for \quad i = 1, \ldots, k \\ 0 & for \quad i = k+1, \ldots, K \end{cases}, \tag{12}$$

and

$$q_i^* = \begin{cases} 0 & for \quad i = 0 \\ \dfrac{\dfrac{w_i}{B_i \left.\dfrac{dp_i(z_i)}{dz_i}\right|_{z_i = z_i^*}}}{\sum_{j=1}^{k} \dfrac{w_j}{B_j \left.\dfrac{dp_j(z_j)}{dz_j}\right|_{z_j = z_j^*}}} & for \quad i = 1, \ldots, k \\ 0 & for \quad i = k+1, \ldots, K \end{cases} \tag{13}$$

The equilibrium payoff Op^* to the Operator is given implicitly by

$$w = \sum_{i=1}^{k} w_i p_i^{-1}\left(\frac{d_i - Op^*}{A_i}\right), \tag{14}$$

and it satisfies the condition

$$d_{k+1} \le Op^* \le d_k \,. \tag{15}$$

The equilibrium payoff In^* to the Inspector is given by

$$In^* = \sum_{i=1}^{k} \left(-c_i + B_i \frac{d_i - Op^*}{A_i}\right) q_i^* \,, \tag{16}$$

where q_i^* and Op^* are given by (13) and (14).

(ii) For $k = K + 1$, i.e., with (10) and (11) for

$$\sum_{i=1}^{K} w_i p_i^{-1}\left(\frac{d_i}{A_i}\right) < w \,, \tag{17}$$

the set of equilibrium strategies of the Inspector is, for all $i = 1, \ldots, K$, given by

$$z_i^* \ge p_i^{-1}\left(\frac{d_i}{A_i}\right) \quad with \quad w = \sum_{i=1}^{K} w_i z_i^* \,. \tag{18}$$

The equilibrium strategy of the Operator is $q_0^* = 1$ and $q_i^* = 0$ for all $i = 1, \ldots, K$, i.e., legal behaviour of the Operator. The payoffs to both players are zero.

Proof. 1) We show that the inequalities in (11) cover the whole parameter space. The proof goes along the same lines as in the proof of Theorem 6.2 in [4]: We show that, for given values of d_i/A_i, $i = 2, \ldots, k$, the inequalities in (11) cover all values of d_1/A_1. For $2 \le k \le K$ and with (7), both inequalities in (11) are equivalent to

$$\frac{d_{k+1}}{A_1} + p_1\left(\frac{w}{w_1} - \sum_{i=2}^{k} \frac{w_i}{w_1} p_i^{-1}\left(\frac{d_i - d_{k+1}}{A_i}\right)\right)$$
$$\le \frac{d_1}{A_1}$$
$$< \frac{d_k}{A_1} + p_1\left(\frac{w}{w_1} - \sum_{i=2}^{k-1} \frac{w_i}{w_1} p_i^{-1}\left(\frac{d_i - d_k}{A_i}\right)\right)$$

Thus, we have to show that

$$\frac{d_{k+1}}{A_1} + p_1\left(\frac{w}{w_1} - \sum_{i=2}^{K} \frac{w_i}{w_1} p_i^{-1}\left(\frac{d_i - d_{k+1}}{A_i}\right)\right)$$
$$< \frac{d_k}{A_1} + p_1\left(\frac{w}{w_1} - \sum_{i=2}^{k-1} \frac{w_i}{w_1} p_i^{-1}\left(\frac{d_i - d_k}{A_i}\right)\right).$$

This is equivalent to

$$p_1\left(\frac{w}{w_1} - \sum_{i=2}^{k} \frac{w_i}{w_1} p_i^{-1}\left(\frac{d_i - d_{k+1}}{A_i}\right)\right) - p_1\left(\frac{w}{w_1} - \sum_{i=2}^{k-1} \frac{w_i}{w_1} p_i^{-1}\left(\frac{d_i - d_k}{A_i}\right)\right) < \frac{d_k - d_{k+1}}{A_1}. \tag{19}$$

We show that the left-hand side of (19) is less than zero and hence, as the right-hand side is by (10) larger than zero, that the inequality holds. By (10) we have $d_k - d_{k+1} > 0$ as well as $d_i - d_{k+1} > d_i - d_k$ and thus, the monotonicity of $p_i^{-1}()$ and $w_i > 0, i = 1, \ldots, K$, implies

$$\sum_{i=2}^{k} \frac{w_i}{w_1} p_i^{-1}\left(\frac{d_i - d_{k+1}}{A_i}\right) > \sum_{i=2}^{k-1} \frac{w_i}{w_1} p_i^{-1}\left(\frac{d_i - d_{k+1}}{A_i}\right)$$
$$> \sum_{i=2}^{k-1} \frac{w_i}{w_1} p_i^{-1}\left(\frac{d_i - d_k}{A_i}\right),$$

which implies that the left-hand side of (19) is less than zero. For $k = 1$ the second inequality in (11) is equivalent to

$$\frac{d_1}{A_1} \ge \frac{d_2}{A_1} + p_1\left(\frac{w}{w_1}\right),$$

and for $k = K + 1$, the first inequality in (11) is, because of $k = K + 1$, equivalent to

$$\frac{d_1}{A_1} < p_1\left(< \frac{w}{w_1} - \sum_{i=2}^{K} \frac{w_i}{w_1} p_i^{-1}\left(\frac{d_i}{A_i}\right)\right),$$

which completes the first part of the proof.

2) We show that (15) holds for $1 \le k \le K$. Assume $Op^* > d_k$. With (14) this implies

$$w = \sum_{i=1}^{k} w_i p_i^{-1}\left(\frac{d_i - Op^*}{A_i}\right) < \sum_{i=1}^{k} w_i p_i^{-1}\left(\frac{d_i - d_k}{A_i}\right)$$
$$= \sum_{i=1}^{k-1} w_i p_i^{-1}\left(\frac{d_i - d_k}{A_i}\right),$$

which is a contradiction to the first inequality in (11). Assume $Op^* < d_{k+1}$. Then (14) implies

$$w = \sum_{i=1}^{k} w_i p_i^{-1}\left(\frac{d_i - Op^*}{A_i}\right) > \sum_{i=1}^{k} w_i p_i^{-1}\left(\frac{d_i - d_{k+1}}{A_i}\right),$$

which is a contradiction to the second inequality in (11).

3) Because the functions $p_i(z_i)$ are monotone increasing and $p_i(0) = 0$ (12) is equivalent to

$$p_i(z_i^*) = \begin{cases} \frac{d_i - Op^*}{A_i} & for \quad i = 1, \dots, k \\ 0 & for \quad i = k+1, \dots, K \end{cases}. \tag{20}$$

Note that $p_i(z_i^*) \ge 0$ is equivalent to $d_i \ge Op^*$ for all $i = 1, \dots, k$ which holds because of (10) and (15), and $p_i(z_i^*) \le 1$ is equivalent to $Op^* \ge d_i - A_i$ which holds because of $Op^* > 0 \ge d_i - A_i = -b_i$ for all $i = 1, \dots, k$. Thus, we have, with (2), (6), (10) and (20),

$$\begin{aligned} Op(\boldsymbol{z}^*, \boldsymbol{q}) &= \sum_{i=1}^{K} \left(d_i - A_i p_i(z_i^*)\right) q_i \\ &= Op^* \sum_{i=1}^{k} q_i + \sum_{i=k+1}^{K} d_i q_i \\ &\le Op^* \sum_{i=1}^{k} q_i + \sum_{i=k+1}^{K} Op^* q_i \\ &\le Op^* \sum_{i=1}^{K} q_i \le Op^* \end{aligned}$$

for all $\boldsymbol{q} \in Q$, i.e., the second inequality in (9) is satisfied.

Before we derive the Inspector's equilibrium strategy, we note that for the Operator's equilibrium strategy we must have $q_i^* = 0$ for all $i = k+1, \dots, K$: Suppose $q_j^* > 0$ for a facility $j, j = k+1, \dots, K$, then the Inspector would have to allocate some of the inspection effort in facility j. However, because of $z_i^* = 0$ for all $i = k+1, \dots, K$, he does not allocate inspection effort in facility j, we must have $q_i^* = 0$ for all $i = k+1, \dots, K$.

Using the first inequality in (9), we determine $\boldsymbol{q}^*$ such that $In(\boldsymbol{z}, \boldsymbol{q}^*)$ is maximized with respect to $\boldsymbol{z}$, and we apply the Lagrange formalism. Using the Lagrange function $L(\boldsymbol{z}, \boldsymbol{q})$ given by

$$L(\boldsymbol{z}, \boldsymbol{q}) = In(\boldsymbol{z}, \boldsymbol{q}) + \lambda\left(w - \sum_{j=1}^{k} w_j z_j\right), \tag{21}$$

we determine $\boldsymbol{q}^*$ such that for all $i = 1, \dots, k$

$$\frac{\partial}{\partial z_i} L(\boldsymbol{z}, \boldsymbol{q}^*)\bigg|_{\boldsymbol{z}=\boldsymbol{z}^*} = 0\,,$$

where the Lagrange parameter λ is determined with the help of the normalization of the $\boldsymbol{q}^*$. Using (6) and (21), the condition

$$\frac{\partial}{\partial z_i} L(\boldsymbol{z}, \boldsymbol{q}^*) = B_i \frac{dp_i(z_i)}{dz_i} q_i^* - \lambda w_i = 0$$

implies for all $i = 1, \dots, k$

$$q_i^* = \lambda w_i \Big/ \left(B_i \frac{dp_i(z_i)}{dz_i}\right),$$

and leads, by using the normalization in (2), to (13). The condition $0 \le q_i^* \le 1$ for all $i = 0, 1, \dots, K$ is obvious, and we even have $0 < q_i^*$ for all $i = 1, \dots, k$. The Hessian $HL(\boldsymbol{z}, \boldsymbol{q})$ of the Lagrange function $L(\boldsymbol{z}, \boldsymbol{q})$ is a diagonal matrix:

$$HL(\boldsymbol{z}, \boldsymbol{q}) = \begin{pmatrix} B_1 \frac{d^2 p_1(z_1)}{dz_1^2} q_1 & 0 & \cdots & 0 \\ 0 & B_2 \frac{d^2 p_2(z_2)}{dz_2^2} q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \frac{d^2 p_k(z_k)}{dz_k^2} q_k \end{pmatrix}.$$

Therefore, the eigenvalues of the Hessian $HL(\boldsymbol{z}, \boldsymbol{q})$ are the diagonal elements. Because $B_i > 0$ and $p_i(z_i)$ are all assumed to be strictly concave functions (see (7)), all eigenvalues of the Hessian $HL(\boldsymbol{z}, \boldsymbol{q}^*)$ are smaller than zero, and thus, the Hessian is negative definite. Therefore, $L(\boldsymbol{z}, \boldsymbol{q}^*)$ is a concave function and Theorem B in [10] implies that $\boldsymbol{z}^*$ is even (because of the concavity of $In(\boldsymbol{z}, \boldsymbol{q}^*)$) a global maximum of $In(\boldsymbol{z}, \boldsymbol{q}^*)$ with respect to $\boldsymbol{z}$.

4) For $k = K + 1$ we see immediately that (18) and $q_0^* = 1$ satisfy the equilibrium conditions (9). This completes the proof.

Three comments on the Theorem. First, as mentioned at the end of section 2, the inspection model in chapter 6 of [4] can be seen as a special case of the inspection model considered in this paper: Put $p_i(z_i) = (1-\beta_i)z_i$ and $w = w_i = 1$ for all $i = 1, \dots, K$. Because we have here $d^2p_i(z_i)/dz_i^2 = 0$ for all $i = 1, \dots, K$, the last condition in (7) is not fulfilled, and thus, the Theorem cannot be applied. Surprisingly, however, the Theorem also covers this case. This can be seen by comparing the Nash equilibrium obtained in case (i) of the Theorem with the Nash equilibrium presented in Theorem 6.2 in [4]. They coincide. Note that because in [4] only illegal behaviour of the Operator is considered, i.e., $q_0 = 0$, case(ii) of the Theorem is not part of Theorem 6.2 in [4].

Second, there are some general features of this solution which are typical for inspection games of this kind, for example, the fact that the equilibrium strategies depend only on the system parameters of the adversaries, or the so-called *cone of deterrence* (18); see [11]. Since, however, it is difficult to discuss more properties of the solution in general terms, we look at an inspection regime based on attribute sampling procedures in section 4 and analyse a time constrained inspection model in section 5.

Third, the Theorem presents a Nash equilibrium but does not address the issue whether there are further Nash equilibria. Indeed, the uniqueness of Nash equilibria in this inspection game is an open question. In section 4 we make a short comment on the uniqueness in case of $K = 2$ facilities and that $p_i(z_i), i = 1,2,$ depends linearly on z_i.

4. First Application: Attribute Sampling

Consider the problem of safeguarding nuclear material in connection with the NPT, in which the role of the Inspector, or player 1, in the model of this paper is played by the IAEA in Vienna: There are K storage facilities for spent nuclear fuel elements in a State (or community of States such as the European Union), operated by an Operator, or player 2. The i-th storage facility contains N_i fuel elements, $i = 1, \dots, K$, the inspection of one of which requires the effort w_i. Thus, if n_i items in the i-th facility are verified, the total inspection effort is

$$w = \sum_{i=1}^{K} w_i n_i \,. \tag{22}$$

For $w_i = w_1, i = 2, \dots, K$, (22) means that the total number of elements to be verified is fixed. Of course, the number $n_i, i = 1, \dots, K$, of verified items in facility i is a nonnegative integer by their very nature. To be able to apply our Theorem, we have to consider n_i as continuous variable. In the applications we have in mind, the n_i may go into the hundreds, therefore, we assume that in these cases the error is small, if nonnatural n_i^* are rounded to natural ones such that the boundary condition is maintained. In the second application (see section 5) there does not exist such a problem.

Furthermore, we assume that in the sense of the NPT it is necessary to detect the diversion of at least one fuel element in one of the facilities. Let us first consider the case that, in order to acquire a so-called *significant quantity* of nuclear material (see [5]), just one fuel element needs to be diverted. The diversion strategy involves replacing the removed fuel element by a dummy.

If the diversion takes place in facility i, then the conditional detection probability (see section 2) in case of drawing without replacement is for all $i = 1, \dots, K$, given by

$$p_i = 1 - \frac{\binom{N_i - 1}{n_i}}{\binom{N_i}{n_i}} = \frac{n_i}{N_i} \,. \tag{23}$$

Because we have all $i = 1, \dots, K$

$$p_i(0) = 0, \quad \frac{dp_i(n_i)}{dn_i} = \frac{1}{N_i} > 0 \quad \text{and} \quad \frac{d^2p_i(n_i)}{dn_i^2} = 0\,,$$

(7) is fulfilled except the last condition. It can be shown, however, that also in this specific case the results of the Theorem are valid; see the first comment after the Theorem. Because (23) is equivalent to $n_i = N_i p_i$ for $i = 1, \dots, K$, we get in case of $1 \le k \le K$ by (12) through (16) the equilibrium strategy of the Inspector

$$n_i^* = \begin{cases} N_i \dfrac{d_i - Op^*}{A_i} & for \quad i = 1, \dots, k \\ 0 & for \quad i = k+1, \dots, K \end{cases}, \tag{24}$$

where k according to (11) is given by

$$\sum_{i=1}^{k-1} w_i N_i \frac{d_i - d_k}{A_i} < w \le \sum_{i=1}^{k} w_i N_i \frac{d_i - d_{k+1}}{A_i}\,,$$

and where with (14) the equilibrium payoff Op^* to the Operator is given by

$$w = \sum_{i=1}^{k} w_i N_i \frac{d_i - Op^*}{A_i}\,.$$

The equilibrium strategy of the Operator is

$$q_i^* = \begin{cases} 0 & for \quad i = 0 \\ \dfrac{w_i N_i}{B_i} \dfrac{1}{\sum_{j=1}^{k} \frac{w_j N_j}{B_j}} & for \quad i = 1, \dots, k \\ 0 & for \quad i = k+1, \dots, K \end{cases}, \tag{25}$$

and the equilibrium payoff In^* to the Inspector is

$$In^* = \sum_{i=1}^{k} \left(-c_i + B_i \frac{d_i - Op^*}{A_i} \right) q_i^* \,, \tag{26}$$

where q_i^* is given by (25). For $k = K + 1$ and with (17) the condition for legal behaviour of the Operator is

$$\sum_{i=1}^{K} w_i N_i \frac{d_i}{A_i} < w, \tag{27}$$

and with (18) the set of equilibrium strategies of the Inspector is for all $i = 1, \dots, K$, given by

$$n_i^* \geq N_i \frac{d_i}{A_i} \quad \text{and} \quad w = \sum_{i=1}^{K} w_i n_i^* \,. \tag{28}$$

At this point we make a remark on the uniqueness of the Nash equilibria given by the Theorem. For $K = 2$ facilities it can be shown that (24) through (28) represent the only Nash equilibrium of the game. Also, it can be shown that in case of legal behaviour of the Operator, i.e., (27) holds, (24) is not an equilibrium strategy of the Inspector for $k = 1$ whereas this is so for $k = 2$.

Now we assume that, in order to acquire a significant quantity of nuclear material, it is necessary to divert not one but two fuel elements, again by replacing them by dummies. If the diversion takes place in facility i, then the conditional detection probability (see section 2) is, for all $i = 1, \dots, K$, given by

$$1 - \frac{\binom{N_i - 2}{n_i}}{\binom{N_i}{n_i}} = 1 - \left(1 - \frac{n_i}{N_i}\right)\left(1 - \frac{n_i}{N_i - 1}\right) \,,$$

or, for our purposes in case of $N_i \gg 1$,

$$p_i = 1 - \left(1 - \frac{n_i}{N_i}\right)^2 . \tag{29}$$

(7) is fulfilled, because

$$p_i(0) = 0, \quad \frac{dp_i(n_i)}{dn_i} = \frac{2}{N_i}\left(1 - \frac{n_i}{N_i}\right) > 0 \quad \text{and}$$

$$\frac{d^2 p_i(n_i)}{dn_i^2} = -\frac{2}{N_i^2} < 0 \,.$$

From (29) we get $n_i = N_i\left(1 - \sqrt{1 - p_i}\right)$ for all $i = 1, \dots, K$, and thus, for $1 \leq k \leq K$ we get from (12) through (16) the equilibrium strategy of the Inspector

$$n_i^* = \begin{cases} N_i \left(1 - \sqrt{1 - \frac{d_i - Op^*}{A_i}}\right) & for \quad i = 1, \dots, k \\ 0 & for \quad i = k+1, \dots, K \end{cases} \tag{30}$$

where k according to (11) is given by

$$\sum_{i=1}^{k-1} w_i N_i \left(1 - \sqrt{1 - \frac{d_i - d_k}{A_i}}\right) < w \leq \sum_{i=1}^{k} w_i N_i \left(1 - \sqrt{1 - \frac{d_i - d_{k+1}}{A_i}}\right),$$

and where with (14) the equilibrium payoff Op^* to the Operator is given by

$$w = \sum_{i=1}^{k} w_i N_i \left(1 - \sqrt{1 - \frac{d_i - Op^*}{A_i}}\right).$$

The equilibrium strategies of the Operator are

$$q_i^* = \begin{cases} 0 & for \quad i = 0 \\ \dfrac{w_i N_i / (B_i (N_i - n_i^*))}{\sum_{j=1}^{k} \frac{w_j N_j}{B_j (N_j - n_j^*)}} & for \quad i = 1, \dots, k \\ 0 & for \quad i = k+1, \dots, K \end{cases} \,, \tag{31}$$

and the equilibrium payoff In^* to the Inspector is

$$In^* = \sum_{i=1}^{k} \left(-c_i + B_i \left(1 - \left(1 - \frac{n_i^*}{N_i}\right)^2\right)\right) q_i^* \,,$$

where n_i^* and q_i^* are given by (30) and (31). For $k = K + 1$ and with (17) the condition for legal behaviour is

$$\sum_{i=1}^{K} w_i N_i \left(1 - \sqrt{1 - \frac{d_i}{A_i}}\right) < w \,, \tag{32}$$

and the set of equilibrium strategies of the Inspector is, for all $i = 1, \dots, K$, given by

$$n_i^* \geq N_i \left(1 - \sqrt{1 - \frac{d_i}{A_i}}\right) \quad \text{and} \quad w = \sum_{i=1}^{K} w_i n_i^* \,. \tag{33}$$

Let us compare conditions (27) and (32) for legal behaviour: Because of

$$\frac{d_i}{A_i} > 1 - \sqrt{1 - \frac{d_i}{A_i}} \,, i = 1, \dots, k \,,$$

the inspection effort w according to (27) has to be larger than that according to (32) which is reasonable: In the former case the number of manipulated fuel element is smaller than in the latter, therefore it is more difficult to detect them.

5. Second Application: Time Constrained Inspections

Let us consider next the problem of drug control at a large international seaport. Assume that K ships of varying sizes have arrived from South American ports of origin and are being unloaded. The port Customs Authority has in total T man-hours at its disposal for inspection of the cargoes for concealed drugs. We might model the detection probability for the i-th ship, $i = 1, \dots, K$, as a function of allotted control time as

$$p_i(t_i) = 1 - \exp(-t_i/\lambda_i) \text{ for } t_i \geq 0 \tag{34}$$

with parameters $\lambda_i > 0$. The expected detection time λ_i will increase with the size of the ship (or of its cargo). We have for all $i = 1, \dots, K$

$$0 \leq p_i(t_i) \leq 1, \quad p_i(0) = 0, \quad \frac{dp_i(t_i)}{dt_i} = \frac{1}{\lambda_i}\exp(-t_i/\lambda_i)$$

$$\frac{d^2 p_i(t_i)}{dz_i^2} = -\frac{1}{\lambda_i^2}\exp(-t_i/\lambda_i) < 0 \,,$$

so that (7) is fulfilled. According to our assumptions, we have

$$\sum_{i=1}^{K} t_i = T \,.$$

We assume additionally that, if drugs are actually being smuggled, it is under the control of a single organization, in the following called Smuggler. In section 6 we will sketch the case that there are several independent Smugglers.

From the Theorem we get with $w_i = w_1$ for $i = 2, \dots, K$ and $w/w_1 = T$ the following equilibrium strategies and payoffs. For $1 \leq k \leq K$, the equilibrium strategy of the Customs Authority is by (12)

$$t_i^* = \begin{cases} \lambda_i \,\mathrm{Ln}\left(\dfrac{b_i + d_i}{b_i + Op^*}\right) & for \quad i = 1, \dots, k \\ 0 & for \quad i = k+1, \dots, K \end{cases} \,,$$

where according to (11) k is given by

$$\sum_{i=1}^{k} Ln\left(1 - \frac{d_i - d_{k+1}}{A_i}\right) \leq -T < \sum_{i=1}^{k-1} Ln\left(1 - \frac{d_i - d_k}{A_i}\right),$$

and where the equilibrium payoff Op^* to the Smuggler is given by

$$T = Ln\left(\prod_{i=1}^{k}\left(\frac{b_i + d_i}{b_i + Op^*}\right)^{\lambda_i}\right).$$

Furthermore, the equilibrium strategy of the Smuggler is by (13)

$$q_i^* = \begin{cases} 0 & for \quad i = 0 \\ \dfrac{\dfrac{\lambda_i}{B_i}\dfrac{b_i + d_i}{b_i + Op^*}}{\sum_{j=1}^{k}\dfrac{\lambda_j}{B_j}\dfrac{b_j + d_j}{b_j + Op^*}} & for \quad i = 1, \dots, k \\ 0 & for \quad i = k+1, \dots, K \end{cases} \,,$$

and the equilibrium payoff In^* to the Customs Authority is given by (16). For $k = K + 1$, and with (17), the condition for legal behaviour is

$$\sum_{i=1}^{K} \lambda_i Ln\left(1 + \frac{d_i}{b_i}\right) < T \,, \tag{35}$$

and the set of equilibrium strategies of the Customs Authority is, for all $i = 1, \dots, K$, given by

$$t_i^* \geq \lambda_i Ln\left(1 + \frac{d_i}{b_i}\right) \quad with \quad T = \sum_{i=1}^{K} t_i^* \,. \tag{36}$$

Here, the lower limit for t_i^* is proportional to the expected detection time and increases monotonically with the ratio d_i/b_i, i.e., with the ratio of the smuggler's incentive to its punishment in the event of detection.

6. Summary and outlook

As already mentioned at the end of section 2, in [2] and in [4] the special case $p_i(z_i) = z_i$ for all $i = 1, \dots, K$, is also considered in a $(K + 1)$-person game with K independently acting Operators, each responsible for one facility only.

Without going into all details of sections 2 and 3, this inspection problem can be analysed in the same way as before under the same assumptions for the Inspector: Whereas the Inspector's strategy set and its (unconditional) expected payoff are again given by (1) and the first equation of (6), the strategy set of the i-th Operator, $i = 1, \dots, K$, and the corresponding (unconditional) expected payoff are given by

$$Q_i = \{q_i : 0 \leq q_i \leq 1\} \quad \text{and} \quad Op_i(\boldsymbol{z}, \boldsymbol{q}) := (d_i - A_i p_i) q_i \,.$$

We will not formulate the solution of this $(K + 1)$-person game as a Theorem. We just report that – not surprisingly – the condition for legal behaviour of all K Operators is again given by (17), respectively (27), (32) and (35), and the cone of deterrence, i.e., the set of strategies of the Inspector in case of legal behaviour of all Operators, again by (18), respectively (28), (33) and (36).

We think that these results, together with those presented in the Theorem and of earlier work, e.g., in [11], describe a universally valid structure of the problem of deterring persons, organizations or even States from illegal behaviour by appropriate inspections.

As a future activity we plan to contact an oversea port authority to discuss according to which criteria its inspection resources are distributed and whether the use of the detection probability (34) is appropriate in the time constrained inspection model of section 5.

7. Disclaimer

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